PH399

Title: Defending Logical Relativism – Examples in Computer Science

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1 Introduction

The position of logical relativism generally advocates that the adequacy of a logic is relative to its purpose. On the other hand, we have logical absolutism, which argues that there is a uniquely adequate logic not relative to anything (Russell, 2013). So, the tension here is between uniqueness and variety. In particular, it is whether logic choice is only constrained by pragmatic factors or that we pick out a uniquely adequate logic. In this paper, I defend a model-centric view of logical relativism. I first begin by outlining the definitions, positions held, and give justifications for this model-theoretic view. Afterwards, I discuss pragmatic factors that are relevant to logic choice. I draw upon examples of logics in computer science, with a focus on model checking. I show pragmatic factors can narrow down logic choice and thus supports logical relativism.

Furthermore, I preempt the objection that these pragmatic factors could become irrelevant over time. If one takes the view that there is a uniquely adequate logic, then the other logics would be expressible in it as sublogics. However, I show that this is a problematic view at best as it runs into technical difficulties. Therefore, without further reworking of concepts such as expressiveness, logical absolutism is problematic. Consequently, logical relativism is a defensible view.

2 Background

We can better understand logical relativism by understanding a similar view called logical pluralism. This is a position that can trace back its motivation to the following statement: "In logic, there are no morals. Everyone is at liberty to build his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments" (Carnap, 1934, page 52). For a logical pluralist, there is no single correct logic, but rather many equally correct logics relative to even the same purpose (Russell, 2013). On a similar pluralist conception of logic, I support the view of logical relativism, that whether a logic is adequate depends on its purpose. I aim to show that because of pragmatic constraints, there are choices to be made from a field of logics to find an adequate logic. For instance, if we are concerned with temporal issues such as "when does property x hold", then temporal logic is the adequate logic instead of first order classical logic. If we want to study vagueness, then a logic that allows truth values between 1 and 0 is the adequate logic. Because the properties we are interested in are captured by models (for instance, temporal logics can be understood by transition systems), this form of logical relativism is model-centric.

The opposite view presented is logical absolutism, which argues that there is a uniquely adequate logic for a given purpose. In light of the model-theoretic focus, I present a model-centric version of this view. We would then hold the view that there is only one adequate logic for a given purpose, whose models can capture everything we want. The other logics are merely special cases of these particularly expansive models. I am to show that this position is highly implausible, and preempt arguments in support of logical absolutism. Hence, this paper will have the following structure:

1. My argument: Pragmatic factors constrain logic choice, even when multiple logics can correctly perform a task.

2. Potential Counterargument: Pragmatic factors may fade over time. So, let's pick the most expressive logic.

3. My response: The formal notion of expressiveness is problematic as it fails to capture desirable intuitive relationships between some logics. So, absolutist need to redefine expressiveness or abandon their line of reasoning.

4. Conclusion: Model-centric Logical Relativism is defensible.

I begin by discussing the pragmatic factors as even if we opt to restrict ourselves to - for instance - the study of temporal logic, there are many cases where one logic is adequate compared to another. Pragmatic constraints help pick out the adequate logic. A counter argument could proceed by the assumption that maybe one day these pragmatic factors are of no concern, and a logical absolutist would argue we can then pick out the most expressive logic. These logics would then have special cases which are the sublogics. However, the formal notion of expressiveness is at best mildly problematic and leads to a tension between an intuitively more expressive logic but cannot satisfy our formal definition. Consequently, I think that logical relativism is a defensible position as the absolutist have to redefine expressiveness or change their reasoning.

3 Pragmatic Factors

Let us begin by examining the pragmatic factors that makes a logic adequate to a purpose. This is a less apparent issue when we compare cases between propositional and first order logic for usage in pure mathematics. However, applied logics often have real world considerations which makes one logic adequate and another inadequate. Consider the field of computer science. Generally, there is a trade-off between the complexity and different tasks for a temporal logic. In particular, we will consider a task called model checking. In model checking, one checks whether concrete implementations of the model satisfy certain formal specifications. For instance, we may want to ascertain that for a lift, the door can never open when the lift is moving, and model checking verifies if this property holds for our system. More technically, model checking is an automatic, model-based, and property-based approach to verification for finite state transition systems. This is a process that can be automated, is semantic oriented, and we have to specify the property we want to test, instead of having a system's full behaviour verified for us (Huth and Ryan, 2004). In model checking, we consider models $\mathcal{M}$ that are transition systems. Transition systems can be thought of as a set of states with paths to move from one state to another, and finite transition systems means there is no infinite set of states. Model-checking is crucial in temporal logic since the notion of truth is not fixed in time. Instead, truth is dynamic and depends on which states of the model we are considering. States can be taken as points in time and a statement such as "p is true in the future" is true only if there is such a state that we can transition to where p is true. There are many ways to see given a model $\mathcal{M}$ and a state s_0 whether $\mathcal{M}, s_0 \models \phi$ for a formula ϕ. Generally, model-checking inputs $\mathcal{M}$, ϕ and outputs all states s of $\mathcal{M}$ satisfying ϕ which we then find if s_0 is among s. (Huth and Ryan, 2004). The logics we want to consider for this example are probabilistic to model random events, and so our transition systems are equipped with probabilities meaning that moving from one state to another is not guaranteed, and we are often more concerned about properties being fulfilled in some range of probability. Our successor states of a system are chosen based on a probabilistic distribution and depends only on the current state thus giving us a stochastic process (Baier and Kaoten, 2008).

The task of model checking can be more or less complex. Tasks that are in the same range of complexity can be grouped up to form a complexity class. Complexity concerns itself with how the output scales with the input. For instance, we want to see how long it takes to verify a property as the specification of that property becomes longer in length. Another example is to checking how a sorting algorithm (which considers a some list as an input and produces a decreasing or increasing list as output) does as we vary the input size (length of list). There are many things that we can use as a measurable resource. We could measure the number of communications, gates in a circuit, or processors needed. We restrict ourselves to the most fundamental resources for model checking, namely time. The following lists several important complexity classes (Gasarch, 2015):

- P: The set of all problems solvable by a deterministic Turing Machine in polynomial time.
- NP: The set of all problems whose instance of the answer 'yes' is verifiable in polynomial time.
- EXPTIME: The set of all problems solvable by a deterministic Turing Machine in exponential time.

The pragmatic factor here is the trade-off between different properties and model checking complexity in multiple applications. Our concern is that given current technology, some logics are in a complexity class that cannot be verified in a reasonable amount of time. Hence we need to change our logic or accept the longer-than-ideal verification time. In the following subsections, I will show why the adequacy of a logic is context sensitive when we examine tasks in computer science due to constraints presented by complexity classes.

3.1 Epidemiology

In this section, I show that pragmatic factors can constrain logic choices. One of the applications of logic is to verify models of disease spread. In doing so, we are subject to some limitations. For example, during the spread of a disease, a key element is time. We want to be able to model how a disease spreads in the fastest way possible so that preventive actions can be taken, but we want to make our model comprehensive enough so that it best represents the situation. To verify our model, there are several logics that we can choose from, but time constraints narrow down which is the most adequate logic to choose. We have a range of logics to choose from such as $PCTL$, $PCTL*$, $PMMC$, LTL, CTL. There is computational tree logic (CTL), whose model is a branching tree of potential futures. Picking this logic allows us to capture the possibilities of choosing different future paths. There is also linear time temporal logic (LTL), which allows us to reason about the future of paths (note the subtle difference between CTL and LTL). However, our main focus for this subsection is Probabilistic Computational Tree Logic ($PCTL$), a logic that allows us to reason about the possible evolutions of a system where the transition from one state to another is not guaranteed. In CTL, we are at liberty to pick which states to transition to but $PCTL$ makes our transition based on probability. In this logic, we are able to express statements such as "in 10 steps, there is a 90% change of a system crash". $PCTL$ model checking is used for verifying ecological models and is useful for epidemiology. We may want to check how statements such as "in 2 days, 20 people will be infected" hold in our model (model checking either verifies the statement or shows it is false). In (Dabrik and Scatena, 2010), we are introduced to the agent-centric SIR model. In this model, the population is divided into three categories: S for susceptible, I for infected, and R for recovered and immune. This model considers each individual and their interactions with each other to model how the disease spreads. Each individual is represented by a finite transition system that interacts with each other to represent individuals coming into contact with each other. This essentially allows us to map out agent interactions and how agents can move between the three categories. There is a point to be made concerning the SIR terminology employed. The traditional variant of the model uses differential equations to map out the relation between the three categories. Our SIR model is the agent-based variant, where we are more concerned with specific agents interacting and how that changes their categories. Consequently, in the differential equation model, our concern is with the equational relationship between the categories and the absolute numbers. We are concerned with the relations between agents in the agent-based variant. So far, It seems that $PCTL$ is the appropriate logic for the task since we want to consider the permutations of different transition systems representing agents interacting with each other and how groups of people become infected. $PCTL$ is the adequate logic because it verify properties of the agent-centric SIR model successfully.

In fact, we cannot satisfactorily use some logics such as Linear Temporal Logic (LTL), which does not permit reasoning about branching time, as we want to know what are the possible evolutions of the system. Consequently, LTL is not adequate for the task. However, one could argue that we could have used other logics such as $PCTL*$ and Probabilistic Modal Mu-Calculus ($PMMC$). $PCTL*$ can do what $PCTL$ does and a bit more (by removing some syntactic restrictions) while $PMMC$ has the addition of fix-point operators (which allows us to consider maximum and minimum solutions to $f(x) = x$) that "can deal with both non-terminating behaviour and recursion in its most general form" (Fontaine, 2010). $PMMC$ and $PCTL*$ can perform the model-checking task as we want to in the agent-centric SIR model. However, the adequacy of these logics are limited by the complexity of model checking which makes $PCTL$ the appropriate logic.

The model-checking complexity of $PCTL$ is known to be $O(poly(size(\mathcal{M})) \times n_{max} \times |\psi|)$ for Markov chain $\mathcal{M}$, $PCTL$ formula ψ, and n_{max} being the maximal step bound in a sub-formula $\phi_1 \mathcal{U}^{\leq n} \phi_2$ of ψ (Baier and Kaoten, 2008). Essentially, the complexity of $PCTL$ is in P, with the growth determined in part by the number of steps into the future we want to consider. $PCTL*$ has a complexity class of $EXPTIME$ and $PMMC$ is known to belong at least in the complexity class UP which contains P but is contained by NP. With this in mind, one can see that we are unable to pick other logics such as $PCTL*$ and $PMMC$ because of the higher complexity class they belong to. This is because model checking is susceptible to what we call the State Space Explosion Problem. This is a case where "the number of states of a concurrent system can grow exponentially with the number of processes" (Clark, et. al, 2018) and makes verification time untenable. The best illustration is for an exponential function, say $f(x) = 2^x$. When x is a small number like 3 or 4, calculating $f(x)$ is trivial, but less so when x is 1985830. (Dabrik and Scatena, 2010) already has an answer to what happens if we encounter a State Space Explosion Problem for the agent-centric SIR model. In particular, if the population we model goes beyond the tens and into the hundreds and thousands, then agent-centric SIR model has an infeasible verification time and the proposed remedy is to abandon this variant of the model for the differential equations SIR model. This is because the properties can only be model-checked in the reasonable time of hours for tens of population. However, if we opt for differential equations, we would lose the accuracy of an agent-based system since we cannot map out all evolutions of an ecological model accurately. Another option proposed by (Dabrik and Scatena, 2010) is to use approximate model checking by means of a technique called abstraction. In doing so, we lose the pinpoint accuracy but can bound the errors. If our purpose is to have predictive power over the SIR model's evolution, then $PCTL$ is irreplaceable especially given its verification time. If we replace $PCTL$ with $PCTL*$ or $PMMC$, then we would be limited to even less state spaces and our modelling capacity for agents would be limited to such an extent that using agent-centric SIR might not even be justified as we cannot consider that many agents. Our limit before the verification time becomes untenable would be significantly less than tens of people, potentially capping at a dozen. At that point, modelling only a dozen people is most likely not of great help to epidemiologist. In this sense, we can see that pragmatic factors do limit logic choice as epidemiological model checking balances complexity class and realism of agent-centric SIR models.

The importance of verification time as a measure for adequacy can be further emphasized by the following example. Consider the recent case of Coronavirus spread in Iran. In the span of a few days, the number of cases officially reported has rapidly increased to around 61 cases in February (BBC, 2020). This is one case where an agent-centric SIR model might be appropriate for the first days of the incident, but as the number of agents to model reach the hundreds, epidemiologist are forced to use a different logic or the differential equation SIR model because we want results in hours, not days. However, during the early stages, a matter of hours needed to verify the agent-centric SIR model is a vital capacity that $PCTL$ provides. Using other more expressive logics like $PCTL*$ would have taken too long. Being able to verify chances of infection and spread gives authorities a clearer image on the scale of the disease and make necessary preparations.

Even in the area of epidemiology our logic is not guaranteed to represent the task correctly and there is a strong case for an adequate logic to be a highly context-sensitive even in specific modelling tasks such that there is no one-size-fits-all approach. As discussed in the previous paragraph, $PCTL$ is still constrained by its complexity class. Because of this, we are unable make add-ons to the agent-centric SIR model needed to model other diseases. For instance, we might not be able to model other diseases such as cancer. For one, cancer is a disease that often takes years to develop unlike diseases such as measles (which is one of the diseases agent-centric SIR models quite well due to its short life-span). This means that there are many states in our transition system required to represent the passage of time which in turn increases the number of potential evolutions of the system. Of course, we are able to make each state represent longer time intervals (instead of every 30 minutes, a state represents a month), though we are at risk of sacrificing the resolution of our model. It is most likely that if we want to model cancer, then LTL is a more appropriate logic since as can reason about the future of paths. LTL is adequate since cancer is not something that spreads from human interaction so does not require the expressiveness of $PCTL$ and the lower complexity class of LTL means that model-checking can allow for more states before becoming untenable. We can see that the adequate logics for mode checking are often times determined by their complexity classes, and applications have nuances whereby settling with $PCTL$ for all epidemiology model-checking is not adequate.

3.2 Fix-point Games

However, there are cases where we absolutely need the properties expressible by $PMMC$ despite its complexity class. As mentioned, $PMMC$ adds fix point operators. One key application of this is in games. In particular, Parity Obligation Games (POG) and Two Player Stochastic Reachability Games (TPSRG). Essentially, both games involve two players with a partitioned configuration. Each has a strategy to reach a certain point and movements are determined by probabilities. These are, however, not concepts that can be formulated in $PCTL$. The task of model checking in general is reducible to the study of parity games. However, to study particular types of parity games and how we can reconfigure model checking to these games, we need to observe $PMMC$. One reason that $PMMC$ is adequate here is because we are not constrained by complexity classes. The applications of this strand of $PMMC$ is more theoretical at this point, so the adequacy of $PMMC$ is not reliant on complexity class when we want to understand the properties of POG. There is another formulation of $PMMC$ which we call PMC instead. This particular formulation is of interest because it has a more tangible application. Namely, it can model peer-to-peer file sharing networks and we are able to reason on stronger safety conditions such as "it will never fail to get a the file" (Larsen, et. al, 2016). Again, for the purpose of finding more operators in our logic that can describe key safety features, PMC and $PMMC$ is the adequate logic even if the complexity is much higher. The properties required for peer-to-peer file sharing cannot be done by $PCTL$ because $PCTL$ cannot reason about recursion and so we accept the higher complexity class as this is our only option to model the system.

In short, we see that pragmatic factors and task-specific requirements constrain logic choices. In the case of epidemiology, while $PCTL*$ and $PMMC$ are correct logics in the sense that agent-centric SIR models can be verified in these logics, they are not adequate because of the higher complexity class that makes model-checking infeasible for epidemiology. On the other hand, when we want to understand the theoretical aspects of model checking and how it is equivalent to parity obligation games and understand the relationship between game theory and model checking, complexity class is not an issue hence $PMMC$ is an adequate logic. Additionally, we see that in the case of peer-to-peer file sharing, the complexity class is less of a constraining factors as we prioritise being able to verify safety conditions (and there is less sense of urgency with verification time compared to epidemiology). Returning to the wider debate, the task of model checking demonstrates that the adequate logic is limited by pragmatic factors such as complexity or trying to convert model checking to parity games.

3.3 Taking Stock

In this section, the key idea is that pragmatic factors do limit what is the adequate logic for model checking. Complexity does not matter when we are concerned purely with how model-checking can be converted to a different task such as parity games. On the other hand, we see that for agent-centric SIR model checking, the adequate logic is constrained by time. As a result, which logic we can use for model checking is a balancing act between complexity class and the properties we want to capture. This supports the argument of a logical relativist, as logic choice is not uniquely determined in the task of model checking because we have to observe some pragmatic limitations.

4 A most expressive logic?

We see in the previous section why pragmatic factors determines what is an adequate logic for a task. A logical absolutist could reply that these constraints could be minimised in the future to the point of irrelevance. One could conceive that the advent of a much quicker computing power makes the state space explosion less of an issue. In this case, the absolutist would argue that where we once had to consider complexity classes and other pragmatic factors, we can pick a logic where logics such that $PCTL$ and $PMMC$ are a sublogic and this one logic is more expressive than other logics for that purpose. Let us grant this assumption of pragmatic factors being irrelevant in some future time, so now our only consideration is expressiveness. In the previous section, I briefly discussed $PCTL*$ and $PMMC$ as logics that can do the same thing as $PCTL$ and more. It seems like a fair point to select the most expressive logic (in our example, $PMMC$). However, there is a problem with this. As I will demonstrate, because of the formal definition of expressiveness, there are some intuitively more expressive logics that are technically incomparable. So, the logical absolutist needs to come up with a better definition of expressiveness that overcomes this particular challenge or accept that there is no most expressive logic.

4.1 Expressiveness as morphisms

The first step here is to formally define expressiveness. Based on intuition, one could argue that a logic is more expressive than another if this logic can say things that the other logic cannot say. For instance, consider first order logic and second order logic. We know that second order logic can express the property of being uncountably infinite, while first order logic cannot because of results called the Lowenheim-Skolem theorems which states any model of first order logic is at most countably infinite. Hence, there are statements in second order logic that has no equivalent in first order logic, but all statements in first order logics have an equivalent in second order logic. This sort of intuition is applied by (Castro, et. al, 2015) and (Mossakowski, et. al, 2009). The particular approach used by (Castro, et. al, 2015) is called encoding, which is essentially a morphism between the two logics. A morphism is a translation of the language of one logic into the language of another logic that preserves the entailment relation (defined later). Our approach is to show that there is a scenario where no morphism exists between a more expressive logic to a less expressive logic because there is a statement in the less expressive logic that cannot be translated. This goes against our intuition of expressiveness because we would expect a more expressive logic to be able to say everything a less expressive logic can say. This notion of encoding, at the moment, is the most widely used definition of expressiveness, so is a good starting point. We start with the notion of an Entailment Relation (ER) (Mossakowski, et. al, 2009).

Definition 1 *An ER is* $\mathcal{S} = (S, \vdash)$ *where* S *is a set of sentences and* $\vdash \subseteq \wp(S) \times S$ *for the set of sentences and sentences such that:*

1. *Reflexivity: for any* $\phi \in \{\phi\}$, $\{\phi\} \vdash \phi$,
2. *Monotonicity: if* $\Gamma \vdash \phi$ *and* $\Gamma' \subseteq \Gamma$, *then* $\Gamma' \vdash \phi$,
3. *Transitivity: if* $\Gamma \vdash \phi_i$ *for* $i \in I$, *and* $\Gamma \cup \{\phi \mid i \in I\} \vdash \varphi$, *then* $\Gamma \vdash \varphi$.

where Γ *is a set of sentences.*

For the sake of length, we omit details and start from the point that our logics fulfil the conditions for an ER, and now we can define a morphism between ER's.

Definition 2 *For two ER's* $\mathcal{S}_1 = (S_1, \vdash_1)$ *and* $\mathcal{S}_2 = (S_2, \vdash_2)$, *an ER-morphism* $\alpha : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ *is a function where* $\alpha : S_1 \longrightarrow S_2$ *where* $\Gamma \vdash_1 \phi$ *implies* $\alpha(\Gamma) \vdash_2 \alpha(\phi)$. *If the converse holds true, then the ER-morphism is conservative.*

Having defined the two key concepts. We can define expressiveness as a case where a logic can have logics encodable to it, which leads us to the following definition (Mossakowski, et. al, 2009):

Definition 3 $\mathcal{S}_1$ *is at most as expressive as* $\mathcal{S}_2$, *represented by* $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$, *iff there exists a conservative* $\alpha : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$.

Note that if we have $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$ and $\mathcal{S}_2 \leq^{ER} \mathcal{S}_1$, the two ER's are equally expressive and can be written as $\mathcal{S}_1 \equiv \mathcal{S}_2$. If $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$ but $\mathcal{S}_1 \not\leq^{ER} \mathcal{S}_2$, $\mathcal{S}_2$ is strictly more expressive than $\mathcal{S}_1$ and we write $\mathcal{S}_1 <^{ER} \mathcal{S}_2$.

To better understand how the above concept applies, consider again the case of $PCTL$ and $PMMC$. As mentioned, there is a way to express $PCTL$ "in" $PMMC$. Before showing that, we need to understand the syntax and semantics of our logics in full. The syntax of $PCTL$ is given by(Castro, et. al, 2015):

Definition 4 $PCTL$ *state formulas are defined by:*

$$\phi ::= a \mid \neg a \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \mathbb{P}_J(\varphi)$$

for $a \in AP$ *(atomic propositions),* φ *is a path formula, and* $j = \{>, \geq\} \times [0,1]$. *The path formulas for* $PCTL$ *are defined as follows:*

$$\varphi ::= \bigcirc\phi \mid \phi_1 \mathcal{U} \phi_2 \mid \phi_1 \mathcal{U}^{\leq n} \phi_2$$

for ϕ *a* $PCTL$ *formula,* $\bigcirc$ *the "next" operator,* $\mathcal{U}$ *the "until" operator, and* $\mathcal{U}^{\leq n}$ *the "bounded step" operator for* $n \in \mathbb{N}$. *We note that the three operators are adequate. Note that 'weak release'* $\neg\phi_1 \mathcal{W}(\neg\phi_2 \wedge \neg\phi_1)$ *in* $PCTL$ *is expressible by* $\neg(\phi_2 \mathcal{U} \phi_1)$ *where* $\psi_0 = \varphi_2$ *and* $\psi_{i+1} = \varphi_1 \wedge \bigcirc\varphi_i$ *for* $i \geq 0$.

The semantics of $PCTL$ is given by a Kripke Structure (Castro, et. al, 2015):

Definition 5 *A Kripke Structure* $\mathcal{K}$ *over a set* AP *is a tuple* (S, R, s_0, L) *where:*

1. S *is the nonempty, countable set of states,*
2. R *is a relation* $R \subseteq S \times S$ *such that for every* $s \in S$, $R(s)$ *is finite,*
3. s_0 *is the initial state*

4. L is the labelling function $L : AP \longrightarrow 2^S$

Furthermore, we define a Markov Chain (Castro, et. al, 2015):

Definition 6 *A Markov Chain over AP is a tuple* $\mathcal{M} = (\mathcal{K}, P)$ *where:*

1. $\mathcal{K}$ *is a Kripke Structure,*
2. P *is a function* $P : R \mapsto (0, 1]$ *so for every* $s \in S$, $\Sigma_{(s,s') \in R} P(s, s') = 1$

Our semantics is defined over a Markov Chain as follows (Baier and Kaoten, 2015):

Definition 7 *For* $\mathcal{M} = (\mathcal{K}, P)$, $a \in AP$, $s \in S$, ϕ *a state formula, and* φ *a path formula, the* $\models$ *relation is defined by:*

1. $s \models a$ *iff* $a \in L(s)$
2. $s \models \neg\phi$ *iff* $s \not\models \phi$
3. $s \models \phi_1 \wedge \phi_2$ *iff* $s \models \phi_1$ *and* $s \models \phi_2$
4. $s \models \mathbb{P}_J(\varphi)$ *iff* $P(s \models \varphi) \in J$

Here, $P(s \models \varphi) = P_s\{\pi \in Paths(s) \mid \pi \models \varphi\}$ *for a path* π *in the set of paths that start at state* s, $Path(s)$*. A path from state* s *is a sequence of states* $s, s_1, s_2, ...$ *starting from* s*.*

Given a path π *in* $\mathcal{M}$*, then the* $\models$ *relation is defined by:*

1. $\pi \models \bigcirc\phi$ *iff* $\pi[1] \models \phi$
2. $\pi \models \phi_1 \mathcal{U} \phi_2$ *iff* $\exists j \geq 0$ *such that* $(\pi[j] \models \phi_2 \wedge (\forall 0 \leq k < j, \pi[k] \models \phi_1))$
3. $\pi \models \phi_1 \mathcal{U}^{\leq n} \phi_2$ *iff* $\exists 0 \leq j \leq n.\ (\pi[j] \models \phi_2 \wedge (\forall 0 \leq k < j.\ \pi[k] \models \phi_1))$

where $\pi[i]$ *denotes the* $(i+1)$*-th state of a path* $\pi = s_0, s_1, ...$

Having explicated $PCTL$, we now consider the syntax and semantics of $PMMC$. Recall that $PMMC$ adds fix-point operators to $PCTL$. The basic formula syntax of $PMMC$ is given below (Castro, et. al, 2015):

Definition 8 *PMMC qualitative formulas are defined by*

$$\varphi ::= a \mid \neg p \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid [\phi]_J \mid \nu X_i.\varphi \mid \mu X_i.\varphi$$

For $a \in AP$*, variables* $X_i \in \mathcal{V} = \{X_0, X_1, ...\}$*, and* $J ::= \{>, \geq\} \times [0, 1]$*. We define PMMC quantitative formulas by:*

$$\phi ::= \varphi \mid X_i \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \diamond\phi \mid \Box\phi \mid \bigcirc\phi \mid \nu X_i.\phi \mid \mu X_i.\phi$$

Qualitative PMMC formulas get their result in either 1 or 0 while quantitative formulas get their result from 0 to 1. A variable X_i is bound in $\sigma X_i.\varphi(X_i)$ for $\sigma \in \{\nu, \mu\}$. An unbound variable is free and a formula is qualitative when it has no free variables. We are now able to define the semantics of $PMMC$, done over a Markov Chain with respect to an interpretation ρ. $\rho : \mathcal{V} \mapsto (S \mapsto [0, 1])$

associates a function from states to a value in $[0, 1]$ with each variable in a formula.

We use a fixed point characterisation of $PMMC$, which the set of states $s \in S$ satisfying a formula ϕ or φ as $[\![\phi]\!]$ or $[\![\varphi]\!]$. Our semantics is given below (Castro, et. al, 2015):

Definition 9 *For a formula ψ over a Markov Chain $\mathcal{M}$ and ρ, we define $[\![\psi]\!]^{\rho}_{\mathcal{M}}$ as follows:*

1. $[\![p_i]\!]^{\rho}_{\mathcal{M}} = \mathcal{X}_{L(p_i)}$
2. $[\![\neg p_i]\!]^{\rho}_{\mathcal{M}} = 1 - \mathcal{X}_{L(p_i)}$
3. $[\![X]\!]^{\rho}_{\mathcal{M}} = \rho(X)$
4. $[\![\psi_1 \vee \psi_2]\!]^{\rho}_{\mathcal{M}} = max([\![\psi_1]\!]^{\rho}_{\mathcal{M}}, [\![\psi_2]\!]^{\rho}_{\mathcal{M}})$
5. $[\![\psi_1 \wedge \psi_2]\!]^{\rho}_{\mathcal{M}} = min([\![\psi_1]\!]^{\rho}_{\mathcal{M}}, [\![\psi_2]\!]^{\rho}_{\mathcal{M}})$
6. $[\![\bigcirc\psi]\!]^{\rho}_{\mathcal{M}} = \lambda s.\Sigma'_s P(s, s')[\![\psi]\!]^{\rho}_{\mathcal{M}}(s')$
7. $[\![\diamond\psi]\!]^{\rho}_{\mathcal{M}} = \lambda s.max_{(s,s') \in RP(s,s')}[\![\psi]\!]^{\rho}_{\mathcal{M}}(s')$
8. $[\![[\psi]_J]\!]^{\rho}_{\mathcal{M}} = ([\![\psi]\!]^{\rho}_{\mathcal{M}}(s)J?1 : 0)$
9. $[\![\Box\psi]\!]^{\rho}_{\mathcal{M}} = \lambda s.min_{(s,s') \in RP(s,s')}[\![\psi]\!]^{\rho}_{\mathcal{M}}(s')$
10. $[\![\mu X.\psi]\!]^{\rho}_{\mathcal{M}} = lfp([\![\psi]\!]^{\rho[f/X]}_{\mathcal{M}})$
11. $[\![\nu X.\psi]\!]^{\rho}_{\mathcal{M}} = gfp([\![\psi]\!]^{\rho[f/X]}_{\mathcal{M}})$

where lfp stands for 'least fixed point', gfp for 'greatest fixed point', $f : S \mapsto [0, 1]$, $\mathcal{X}_{L(p_i)}$ is a function where $\mathcal{X}_{L(p_i)}(s) = 1$ for $s \in L(p_i)$ and $\mathcal{X}_{L(p_i)}(s) = 0$ for $s \notin L(p_i)$ where $L(p_i) \subseteq S$. $\rho[f/X]$ is interpreted as the function that associates a function f with X and $\rho(Y)$ with all $Y \neq X$.

Having now defined the syntax and semantics, we note that there is a proof that $PMMC$ is more expressive. This is done by constructing a morphism that meets the criteria of an ER-morphism. We construct such a morphism below (Castro, et. al, 2015):

Definition 10 *Let $\alpha(\psi)$ be the formula in $PMMC$ obtained by transforming a $PCTL$ formula ψ recursively where:*

1. $\alpha(p_i) = p_i$
2. $\alpha(\neg p_i) = \neg p_i$
3. $\alpha(\phi_1 \wedge \phi_2) = \alpha(\phi_1) \wedge \alpha(\phi_2)$
4. $\alpha(\phi_1 \vee \phi_2) = \alpha(\phi_1) \vee \alpha(\phi_2)$

5. $\alpha(\phi_1 \mathcal{U} \phi_2) = \mu X.\alpha(\phi_2) \vee (\alpha(\phi_1) \wedge \bigcirc X)$

6. $\alpha(\mathbb{P}_J \varphi) = [\alpha(\varphi)]_J$

7. $\alpha(\bigcirc \phi) = \bigcirc \alpha(\phi)$

8. $\alpha(\phi_1 \mathcal{W} \phi_2) = \nu X.\alpha(\phi_2) \wedge (\alpha(\phi_1) \vee \bigcirc X)$

Hence, by definition of expressiveness, we have $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$ where the ER $\mathcal{S}_2$ is $PMMC$ and the ER $\mathcal{S}_1$ is $PCTL$. However, we cannot construct an ER-morphism such that $\mathcal{S}_2 \leq^{ER} \mathcal{S}_1$. We can construct a formula in $PMMC$ that cannot be transformed to a sentence in $PCTL$. An example is the sentence $\nu X.p_i \wedge \diamond \diamond \phi$ where there is no corresponding $PCTL$ sentence under any transformation (Kozen, 1983). Because of this, we have $\mathcal{S}_1 \not\equiv \mathcal{S}_2$ since $\mathcal{S}_2 \not\leq^{ER} \mathcal{S}_1$, therefore $\mathcal{S}_1 <^{ER} \mathcal{S}_2$. So, $PMMC$ is strictly more expressive than $PCTL$.

So far, this seems like a fairly convincing construction of expressiveness. It has successfully shown that $PMMC$ is more expressive than $PCTL$ and preserved our intuition of expressiveness. However, this is where we run into issues regarding the formal definition. One can show that there is a construction of $PMMC$ such that there is no morphism from this logic to $PCTL$. This seems problematic. We know that $PMMC$ is more expressive than $PCTL$ because $PMMC$ reasons about fix-points in addition to non-deterministic branching-time that is described by $PCTL$. Not only that, they are both extensions of their non-probabilistic versions called CTL and MMC. We can prove that CTL is a sublogic of MMC, so we should be able to maintain this relationship even if we add probabilities to both logics. This formulation of $PMMC$ (hereon PMC for ease of reference) has no translation for particular segments of $PCTL$. PMC has an "extended n-ary next-state modality ((in-)equational modality) was introduced: $[\langle x \rangle \phi_1, \langle y \rangle \phi_2 : x + y \leq 0.7]$ describes that the probabilities x and y of reaching next-states satisfying ϕ_1 and ϕ_2 respectively must satisfy the constraint $x + y \leq 0.7$. This modality allows one to encode complex linear constraints on probabilities" (Larsen, et. al, 2016). So, it seems odd that the addition of probabilistic inequalities affect expressiveness in such a way that it is unable to encode some sentences in $PCTL$, but can express more properties than $PCTL$. PMC is still interpreted over a markov chain and block-semantics is introduced, which are additions to semantics of $PMMC$ by imposing restrictions on a sequence of blocks. However, this makes it such that we cannot express modalities such as the $PCTL$ 'probabilistic Until' $(\phi_1 \mathcal{U} \phi_2)$ (Larsen, et. al, 2016). While (Larsen, et. al, 2016) notes that we can still approximate the $PCTL$ 'probabilistic Until' operator to some degree of accuracy given a finite model in PMC, the result is unsatisfying when we define expressiveness in such a rigorous way where approximation is not enough and encoding is demanded.

One could argue that some logics are incomparable. Take epistemic logic and temporal logics. They are extensions of first order logic, but each can express wildly different things (one of knowledge, and the other of time), neither of which should be more expressive than the other because they aim to capture different notions. We can then aim to restrict what sorts of ER we want to consider when we define expressiveness (though it is not clear if we can make an objective criteria for this). This is an open ended proposal, but for $PCTL$ and PMC, we have not added and removed different operators, but rather augmented $PCTL$ only by addition of fixed point operators while keeping the rest of $PCTL$ constant. We still want to work with the notion that truth is not fixed in time, and have not shifted to a different region such as reasoning about knowledge. Hence these temporal logics should belong in the same category, and it is fair to claim that PMC should be comparable and based on our intuition, formally more expressive than $PCTL$ by addition of operators. So, it seems that even when our definitions of expressiveness is a rigorous one, there are cases where something informally more expressive actually cannot subsume properties of a less expressive logic. I conclude that this leaves logical absolutist in a difficult spot. One possible approach is to redefine expressiveness in such a way that can formally capture the intuition that PMC is more expressive than $PCTL$. However, it is not certain that this works for the multitudes of other logics and is no guaranteed counter-argument.

4.2 Taking Stock

In this section, we have seen that an approach to just picking the most expressive logic encounters a significant issue. We want to capture the intuition that expressiveness is one logic can say something that another logic cannot, while being able to say everything a less expressive logic can say. To this end, we have worked with the most commonly used formal concept of expressiveness. Called encoding in some papers, it is essentially a morphism from one logic to another. This seems to formally capture our notion of expressiveness. However, we encounter a problem when $PCTL$ is encodable to $PMMC$, but is not encodable into PMC. Intuitively, PMC says more things than $PCTL$, but PMC can only approximate the 'probabilistic Until' operator in $PCTL$. Hence, this leaves the logical absolutist in a difficult spot. They have to present a conception of formal expressiveness that can account for this oddity, or abandon the idea that there is a most expressive logic such that logic choice has a unique outcome.

5 Conclusions

To conclude, I have defended the model-centric version of logical relativism. I began by first outlining the positions relevant to this paper, then proceeded to demonstrate that there are pragmatic factors that makes some logic more adequate than others and that logic choice is dependent on pragmatic factors. To this end, I considered the task of model checking in general and considered two activities within it. In the case of epidemiology, $PCTL$ is the adequate logic because time is a significantly constraining pragmatic factor. On the other activity, I note that complexity is less relevant, and we are interested in learning about the conversion of model checking tasks to parity games instead so we want expressiveness to capture different notions convertible to game theoretic ones. In this latter task, we then have a different adequate logic. Even if one argues that pragmatic factor such as time may be remedied by technological advances, we cannot simply pick out the most expressive logic for our tasks. As it turns out, the most commonly used formal notion of expressiveness can contradict our intuitions. A logic such as PMC should be more expressive than $PCTL$ due to the addition of fixed point operators and keeping everything else unchanged, but it turns out that this is not formalizeable as PMC can only approximate a particular operator in $PCTL$. Hence, the position of logical relativism is defensible.

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